Numerical Schemes for Variational Inequalities arising in International Asset Pricing

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1. Introduction

We develop a continuous time model of international asset pricing in a two-country framework with political risks. The structure of the model is similar to Dumas (1992) except for the inclusion of political risk. Assets are homogeneous except for location, and serve as production inputs as well as consumption goods. International capital markets are fully integrated in the sense that individuals from each country can freely buy and sell claims to assets located in both countries. However, individuals can only consume assets located in their country of residence. Assets can be shipped between countries at a cost and subsequently may be utilized for either production or consumption purposes at their new location.

Political risk enters the model via uncertainty in the drift of the stochastic production process associated with the politically risky country. Fundamentally, political risk represents uncertainty about future government actions which may impact the value of firms and/or the welfare of individuals. If we focus simply on the value of firms, there are still a lot of government actions which can affect firm profitability and the values of securities it issues. Changes in the tax code (or its implementation), price ceilings, local content requirements, quotas on imported inputs, labor law provisions, and numerous other areas of government regulation can affect firm profitability and/or security values. One could argue that all governments exhibit political risk in that there is some uncertainty about their future actions. However, the degree of risk tends to vary dramatically with some governments (countries) viewed as "politically stable" and others as quite risky.

For simplicity, we treat one of our countries as exhibiting no political risk. That is, the drift is known for the production process of assets located in that country. There is still uncertainty about the value of production assets located in that country due to market forces, technology, weather, etc. We associate such uncertainty with a (country specific) brownian motion; however, the drift of the process is known in the politically stable country. In the politically risky (unstable) country, we assume that the expected productivity of assets can take on two values. The lower state can be interpreted as representing the local government's ability to impose a tax or regulation on firms producing in that country which negatively impacts their profitability. Symmetrically, the high state can be interpreted as either a lower tax rate or even a subsidy for local production, perhaps in an indirect form via changing a restrictive regulation.

Furthermore, a negative action can be followed by a positive one and vice versa. Recently, we have seen asset prices and exchange rates (a special type of asset price) yoyoing up and down in response to sequences of government actions, as well as conjectures about future actions. In a rather simplified manner, we are attempting to capture this sort of phenomenon by having the drift in the risky country determined by a continuous time Markov chain which can take one of two possible values. Consequently, the extent of political risk in our model is determined by both the spread between the two drift parameters from the Markov chain and by the transition probabilities.

We formulate this model as a singular stochastic control problem whose states describe the production technology processes in both countries. The collective utility is the value function of this optimization problem and it is characterized as the unique (weak) solution of the associated Hamilton-Jacobi-Bellman (HJB) equation. Because of the presence of the shipping costs and the effects of the Markov chain process, the HJB equation actually turns out to be a *system of Variational Inequalities, coupled through the zeroth order terms, with gradient constraints.* Such problems typically result in a depletion of the state space into regions of idleness and regions where singular controls are exercised. In the context of the model we are developing herein, the singular policies correspond to "lump-sum" shipments from one country to the other.

Similar problems with singular policies arise in a wide range of models in the areas of asset and derivative pricing. They are essentially linked to the fundamental issue of irreversibility of financial decisions in markets with frictions such as transaction or shipping costs or an irreversible loss of an investment opportunity related to unhedgeable risks. Unfortunately, such problems do not have in general smooth solutions, let alone closed form ones. It is therefore imperative to analyze these problems numerically by building accurate schemes for the value function as well as the free boundaries which characterize the singular investment policies.

We undertake this task and we construct a family of numerical schemes for the collective utilities and the equilibrium prices. These schemes have all the desired properties for convergence, namely, *stability, monotonicity and consistency.* They belong to the class of the so-called "time-splitting" schemes which approximate separately — in each half-time iteration — the first- and the second-order derivatives. These schemes are known to be very suitable for the approximation of the solutions of a certain type of second-order nonlinear partial differential equations similar to the ones arising in our model.

Although it is highly simplified, the proposed model captures some of the flavor of an international environment where assets may be exposed to substantially different risks because they are located within the jurisdictions of different governments. In effect, they are different assets and will generally exhibit different prices because of their location. As we shall see, political risk not only influences asset values but also consumption patterns. Furthermore, if we interpret the ratio of the output prices in the two economies as a real exchange rate, then that exchange rate will exhibit sustained deviations from its Purchasing Power Parity (PPP) value.

The paper is organized as follows. In section 2, we describe the basic model and we provide analytic results for the value function. In section 3, we construct the numerical schemes for the value function and the trading policies. In section 4, we interpret the numerical results, and we provide some conclusions and suggestions for extensions of this work.

2. The model and the associated Variational Inequalities.

In this section we describe the international asset pricing model we are going to use in order to study the effects of political risk on international asset prices, consumption and investment behavior across countries.

We concentrate on a simplified two-country model where capital markets are fully integrated in the sense that individuals from each country can own claims to assets located in either country. These assets serve as production inputs and consumption goods. The production technology is stochastic and differs across the two countries; however there is a single technology in each country. One of the two countries is considered "politically unstable" and exhibits political risk. We model the political risk via a continuous time *Markov chain* which affects the rate of return of the stochastic production process in the politically risky country. We assume, for simplicity, that there are only two states for the Markov chain, a low and a high state. As it was pointed out in the introduction, the different states can be interpreted, among other things, as representing the local government's ability to alter the tax rate on firms producing in that country or to alter its subsidizing policy on local production. In this context, the low state corresponds to a high tax rate, with the high state representing either a low tax or perhaps a subsidy.

We denote the goods of the two countries by **X** and **Y** and the production technology processes in the countries $\mathcal{X}$ and $\mathcal{Y}$ by X_t and Y_t respectively. Country $\mathcal{Y}$ is considered to be politically stable and its production process Y_t is modelled as a diffusion process with drift coefficient b and volatility parameter σ_2. Country $\mathcal{X}$ has a production technology process with similar diffusion structure — with volatility parameter σ_1 — but its drift coefficient is affected by a two-state Markov chain, say z_t, which represents the effects of the *political instability*.

Consumption on country $\mathcal{X}$ is denoted by C^x, which includes both consumption of local output and of imports from country $\mathcal{Y}$. Consumption in country $\mathcal{Y}$ is defined in an analogous manner and is denoted by C^y. Cumulative shipments, as of time

t, from country $\mathcal{X}$ to country $\mathcal{Y}$ are denoted by L_t; such shipments (exports from country $\mathcal{X}$) incur proportional shipping costs at a rate λ. In a similar manner, cumulative shipments from country $\mathcal{Y}$ (imports by country $\mathcal{X}$), denoted by M_t incur proportional shipping costs at a rate μ. Without loss of generality, we assume that country $\mathcal{X}$ is charged with the shipping costs.

Using the above definitions, we can write the state processes for the capital stocks in the two countries as

$$dX_t = z_t X_t dt - C_t^x dt + \sigma_1 X_t dW_t^1 - (1+\lambda) dL_t + (1-\mu) dM_t \tag{2.1}$$

$$dY_t = bY_t dt - C_t^y dt + \sigma_2 Y_t dW_t^2 + dL_t - dM_t \tag{2.2}$$

with W_t^1 and W_t^2 being brownian motions on a probability space $(\Omega, \mathcal{F}, P)$ with correlation $\delta \in [-1, 1]$; for this we can take $W_t^2 = \delta W_t^1 + \sqrt{1-\delta^2}\, B_t$ with B_t being a brownian motion independent of W_t^1. The constants σ_1, σ_2 and b are assumed to be positive.

The process z_t is a continuous-time Markov chain with two states z_1 and z_2 such that

$$0 < z_1 < z_2 \tag{2.3}$$

As it was discussed above, the low state z_1 is associated with an unfavorable political state (from the perspective of the production process owners) as opposed to the high state z_2 which represents the favorable political state in country $\mathcal{X}$. We denote by $p_{i,j}$, $i, j = 1, 2$ the transition probabilities of z_t for the above states.

The *collective* (or integrated) *utility payoff* for consumers of both countries over their consumption rates is

$$E \int_0^{+\infty} e^{-\rho t} U(C_t^x, C_t^y) dt.$$

A policy (C_t^x, C_t^y, L_t, M_t) is admissible if it is $\mathcal{F}_t$-progressively measurable — where $\mathcal{F}_t = \sigma((W_s^1, B_s, z_s) : 0 \leq s \leq t)$ — with L_t and M_t being nondecreasing

CADLAG* processes such that $C_t^x \geq 0$, $C_t^y \geq 0$ a.s.,

$$E\int_0^t e^{-\rho s}(C_s^x + C_s^y)ds < +\infty \qquad t \geq 0$$

and the following state constraints are satisfied

$$X_t \geq 0 \text{ and } Y_t \geq 0 \text{ a.e. } t \geq 0. \tag{2.4}$$

The collective consumer function $U : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is assumed to be increasing and concave in both arguments. Also, $U(0, 0) = 0$ and U satisfies

$$U(C^x, C^y) \leq M\Big(\frac{1}{1+\lambda}C^x + (1-\mu)C^y\Big)^\gamma$$

for some constants $M > 0$ and $0 < \gamma < 1$.

We define the collective across-countries value function $V(x, y; z)$ as

$$V(x, y; z) = \sup_{\mathcal{A}_z} E\int_0^{+\infty} e^{-\rho t}U(C_t^x, C_t^y)dt \tag{2.5}$$

The set $\mathcal{A}_z$ is the set of admissible policies (C_t^x, C_t^y, L_t, M_t) which are assumed to satisfy the above measurability and integrability conditions and the state constraints (2.4); ρ is a positive constant which plays the role of a discount factor.

In order to guarantee that V is finite for all $x \geq 0$, $y \geq 0$ and $z = z_1, z_2$, we impose the following restrictions on the market parameters.

First, we define $\sigma, k_i, \hat{\rho}_i$, $i = 1, 2$ as follows:

$$\sigma = \sqrt{\frac{1}{2}\sigma_1^2 - \delta\sigma_1\sigma_2 + \frac{1}{2}\sigma_2^2}\,.$$

If $b > z_2$,

$$\begin{cases} k_1 = \sigma_1^2(1-\gamma) - (1-\gamma)\delta\sigma_1\sigma_2 + (b - z_2) \\ \hat{\rho}_1 = \rho + \frac{1}{2}\gamma(1-\gamma)\sigma_1^2 - \gamma z_2 \end{cases}$$

* A process is CADLAG if it is right-continuous with left limits.

and if $b < z_2$,

$$\begin{cases} k_2 = \sigma_2^2(1-\gamma) - (1-\gamma)\delta\sigma_1\sigma_2 + (z_2 - b) \\ \hat{\rho}_2 = \rho + \frac{1}{2}\gamma(1-\gamma)\sigma_2^2 - \gamma b. \end{cases}$$

We assume that at least one of the following sets of inequalities holds

$$\{b \geq z_2, k_1 > 0, \hat{\rho}_1 > 0\} \quad \text{or} \quad \{b < z_2, k_2 > 0, \hat{\rho}_2 > 0\}, \tag{2.6a}$$

together with the additional related conditions

$$\hat{\rho}_1 > z_2(\gamma - 1) + \frac{\gamma k_1^2}{2\sigma^2(1-\gamma)}, \quad \text{if} \quad b \geq z_2$$

or (2.6b)

$$\hat{\rho}_2 > b(\gamma - 1) + \frac{\gamma k_2^2}{2\sigma^2(1-\gamma)}, \quad \text{if} \quad b < z_2 \ .$$

We continue with some elementary properties of the value function.

Proposition 2.1: The value function V is increasing and jointly concave in the spatial arguments (x, y). Moreover, for fixed z, V is uniformly continuous on $[0, +\infty) \times [0, +\infty)$.

Proof: The monotonicity follows from the fact that for the point $(x+\epsilon, y)$ (respectively $(x, y+\epsilon)$ the set of admissible policies $\mathcal{A}_{z,(x+\epsilon,y)}$ satisfies $\mathcal{A}_{z,(x+\epsilon,y)} \supset \mathcal{A}_{z,(x,y)}$; the latter follows from the monotonicity and concavity of the utility function, the form of the state dynamics and the definition of the value function. These properties, together with the state constraints (2.4) are also used to establish the concavity of the value function. Indeed, if (C_1^x, C_1^y, L_1, M_1) and (C_2^x, C_2^y, L_2, M_2) are optimal policies for the points $(x_1, y_1; z)$ and $(x_2, y_2; z)$, then for $\lambda \in (0, 1)$, the policy $(\lambda C_1^x + (1-\lambda)C_2^x, \lambda C_1^y + (1-\lambda)C_2^y, \lambda L_1 + (1-\lambda)L_2, \lambda M_1 + (1-\lambda)M_2)$ is admissible for $(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2; z)$. For the uniform continuity of the value function on $[0, +\infty) \times [0, +\infty)$ we refer the reader to Proposition 2.2 in Tourin and Zariphopoulou (1994).

Proposition 2.2: Under the growth conditions (2.6a) and (2.6b) and the properties of the utility function U, the value function is well defined on $[0, +\infty) \times [0, +\infty)$ for $z = z_1, z_2$.

The proof appears in Appendix B.

Remark 2.1: Even though we look at the case of linear coefficients in the state equations (2.1) and (2.2), this assumption is by no means restrictive. In fact, all the arguments presented herein can be easily generalized to the case of general coefficients $\sigma_1(X_t)$, $b(Y_t)$ and $\sigma_2(X_t)$ as long as the functions $\sigma_1, b, \sigma_2 : [0, +\infty) \rightarrow [0, +\infty)$ satisfy the conditions: they are Lipschitz concave functions of their argument with $\sigma_1(0) = b(0) = \sigma_2(0)$ and at least one of the σ_i's, $i = 1, 2$ satisfies $\sigma_i(w) > mw$, for $w \geq 0$ and $m > 0$. ** The motivation behind the choice of linear coefficients is only for the sake of simplicity; the methodology is easier to present and also, the numerical schemes are validated for such coefficients.

The classical way to attack problems of stochastic control is to analyze the relevant equation that the value function is expected to solve, namely the Hamilton-Jacobi-Bellman equation. This (HJB) equation is the offspring of the Dynamic Programming Principle and stochastic analysis. When singular policies are allowed, the HJB equation becomes a Variational Inequality with gradient constraints. These constraints are associated to the "optimal direction" of instantaneously moving the optimally controlled state processes. In the context of optimal consumption and investment problems, such situations arise when transaction fees are paid (see, for example, Zariphopoulou (1991), Tourin and Zariphopoulou (1994)). In the problem we study herein, the analysis is more complicated because the drift of the state process X_t is influenced by the fluctuations of the Markov chain z_t. This feature, together with the presence of singular policies results into an HJB equation which

** For a similar problem with general coefficients, we refer the interested reader to Scheinkman and Zariphopoulou (1997).

is actually a *system of Variational Inequalities* coupled through the zeroth order term (see equations (2.7) and (2.8)).

It is a well established fact that if it is known a priori that the value function is smooth, then standard verification results guarantee that it is the unique smooth solution of the HJB equation. Moreover, if first order conditions for optimality apply, then they are sufficient to determine the optimal policies in the so-called feedback formula. (See, for example, Fleming and Soner (1993)). Unfortunately, this is rarely the case. As in our problem, the value function might not be smooth and therefore it is necessary to relax the notion of solutions of the (HJB) equation. These weak solutions are the so-called (constrained) *viscosity solutions* and this is the class of solutions we will be using throughout the paper. In models of optimal investment and consumption with transaction costs, which are a special case of the model described above, this class of solutions was first employed by Zariphopoulou (1992). Subsequently this class of solutions was used among others by Davis, Panas and Zariphopoulou (1993), Davis and Zariphopoulou (1994), Tourin and Zariphopoulou (1994), Shreve and Soner (1994), Barles and Soner (1995), and Pichler (1996). The characterization of V as a *constrained* solution is natural because of the presence of state constraints given by (2.4).

The notion of viscosity solutions was introduced by Crandall and Lions (1983) for first-order equations, and by Lions (1983) for second-order equations.
Constrained viscosity solutions were introduced by Soner (1986) and Capuzzo-Dolcetta and Lions (1987) for first-order equations (see also Ishii and Lions (1990)). For a general overview of the theory we refer to the *User's Guide* by Crandall, Ishii and Lions (1994) and to the book by Fleming and Soner (1993). We provide the definition of constrained viscosity solutions in Appendix A.

The following theorem provides a unique characterization of the value function. Its proof is discused in Appendix B.

Theorem 2.1. *The value function is the unique constrained viscosity solution on*

$[0, +\infty) \times [0, +\infty)$, *of the system of the variational inequalities*

$$\begin{aligned} \min \big\{ & \rho V(x,y;z_1) - \mathcal{L}V(x,y;z_1) - H(V_x(x,y;z_1),\ V_y(x,y;z_1)) \\ & - p_{12}(V(x,y;z_2) - V(x,y;z_1)) - z_1 x V_x(x,y;z_1) \\ & - byV_y(x,y;z_1),\ (1+\lambda)V_x(x,y;z_1) - V_y(x,y;z_1), \\ & - (1-\mu)V_x(x,y;z_1) + V_y(x,y;z_1) \big\} = 0 \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \min \big\{ & \rho V(x,y;z_2) - \mathcal{L}V(x,y;z_2) - H(V_x(x,y;z_2),\ V_y(x,y;z_2)) \\ & - p_{21}(V(x,y;z_1) - V(x,y;z_2)) - z_2 x V_x(x,y;z_2) \\ & - byV_y(x,y;z_2),\ (1+\lambda)V_x(x,y;z_2) - V_y(x,y;z_2), \\ & - (1-\mu)V_x(x,y;z_2) + V_y(x,y;z_2) \big\} = 0 \end{aligned} \tag{2.8}$$

in the class of concave and increasing functions with respect to the spatial argument (x, y). *Here* $\mathcal{L}$ *is the differential operator*

$$\mathcal{L}V = \frac{1}{2}\sigma_1^2 x^2 V_{xx} + \delta\sigma_1\sigma_2 xyV_{xy} + \frac{1}{2}\sigma_2^2 y^2 V_{yy} \tag{2.9}$$

and

$$H(q_1, q_2) = \max_{C^x \geq 0, C^y \geq 0} \big\{ -q_1 C^x - q_2 C^y + U(C^x, C^y) \big\}. \tag{2.10}$$

As it was mentioned earlier, the presence of singular policies leads to a depletion of the state space into regions of three types, namely the "Import to country $\mathcal{Y}$" $(I_{\mathcal{Y}})$, "Export from country $\mathcal{Y}$" $(\mathcal{E}_{\mathcal{Y}})$ and "No-shipping" $(\mathcal{NS})$ regions. The choice for the country $\mathcal{Y}$ to be used as the baseline for the description of the optimal trading rules is arbitrary and does not change the nature of the results. The regions $(I_{\mathcal{Y}})$, $(\mathcal{E}_{\mathcal{Y}})$ and $(\mathcal{NS})$ are related to the optimal shipping rules as follows: if at time t the production technology state (X_t, Y_t) belongs to $\mathcal{NS}$ region, only the consumption processes are used and no shipments take place from one country to the other.

If the state (X_t, Y_t) belongs to the $(I_{\mathcal{Y}})$ (respectively, $\mathcal{E}_{\mathcal{Y}}$) region, it is beneficial to the central planner to import (respectively, export) a shipment from country $\mathcal{X}$ to country $\mathcal{Y}$. In other words, a singular policy—which represents the "lump-sum" shipment—is used to reduce (respectively, increase) the value of X_t in order to increase (respectively, decrease) the value of Y_t and move to a new state, say (X_{t+}, Y_{t+}) which belongs to the boundary of the $(\mathcal{NS})$ and the $(I_{\mathcal{Y}})$ (respectively, $(\mathcal{E}_{\mathcal{Y}})$) region.

No closed-form solutions exist up to date for the free boundaries of the aforementioned $(I_{\mathcal{Y}})$, $(\mathcal{E}_{\mathcal{Y}})$ and $(\mathcal{NS})$ regions. Therefore, it is highly desirable to analyze these boundaries as well as other related quantities, numerically. This is the task we undertake in the next section.

Remark 2.2: In the special case of a collective utility function of the CRRA type, $U(C^x, C^y) = [(C^x)^\gamma + (C^y)^\gamma]$ for $0 < \gamma < 1$, one can show that the value function is homogeneous of degree γ. This fact provides valuable information about the free boundaries which turn out to be straight lines passing through the origin.

We continue this section by presenting some results related to analytic bounds of the value function as well as alternative characterizations of it in terms of a class of "pseudo-collective" value functions. The latter results are expected to enhance our intuition for the economic significance of the proposed pricing model. We only present the main steps of the proofs of these results; the underlying idea is to use the HJB equations (2.7) and (2.8) and interpret them as HJB equations of new pseudo-utility problems. The comparison between the new "pseudo-value functions" and the original value function stems from the uniqueness result in Theorem 2.1 as well as the fact that the pseudo-value functions are viscosity solutions of the associated HJB equations.

To this end, consider the following pairs $(\underline{x}_t, \underline{y}_t)$ and $(\bar{x}_t, \bar{y}_t)$ of state dynamics where $\underline{x}_t$, $\underline{y}_t$, $\bar{x}_t$ and $\bar{y}_t$ solve respectively

$$d\underline{x}_t = z_1 \underline{x}_t dt - \underline{C}^x_t dt + \sigma_1 \underline{x}_t dW^1_t - (1+\lambda) d\underline{L}_t + (1-\mu) d\underline{M}_t \tag{2.11}$$

$$dy_t = by_t dt - \underline{C}_t^{y} dt + \sigma_2 \underline{y}_t dW_t^2 + d\underline{L}_t - d\underline{M}_t \tag{2.12}$$

and

$$d\bar{x}_t = z_2 \bar{x}_t dt - \bar{C}_t^{\bar{x}} dt + \sigma_1 \bar{x}_t dW_t^1 - (1+\lambda) d\bar{L}_t + (1-\mu) d\bar{M}_t \tag{2.13}$$

$$d\bar{y}_t = b\bar{y}_t dt - \bar{C}_t^{\bar{y}} dt + \sigma_2 \bar{y}_t dW_t^2 + d\bar{L}_t - d\bar{M}_t \tag{2.14}$$

where z_1, z_2 are the two states of the process z_t and $\underline{x}_0 = \overline{x}^0 = x$, $\underline{y}_0 = \overline{y}^0 = y$. The above dynamics correspond to the case of deterministic drifts with no effect from the Markov chain.

We define for (2.11), (2.12) and (2.13), (2.14) the sets of admissible policies $\mathcal{A}_{z_1}$ and $\mathcal{A}_{z_2}$ along the same lines as before.

The following result shows that the original value function V is bounded between $\underline{v}$ and $\bar{v}$ with $\underline{v}$ and $\bar{v}$ being respectively the value functions of two international asset pricing models with the original collective utility but with no political risk. More precisely, $\underline{v}$ (respectively $\bar{v}$) is the collective value function for countries $\underline{\mathcal{X}}$ and $\underline{\mathcal{Y}}$ (respectively $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$) with $\underline{\mathcal{X}}$ (respectively $\overline{\mathcal{X}}$) not exhibiting political instability but with a modified mean rate of return in its capital stock. Models of this type were studied by Dumas (1992) in the case of CRRA utilities.

Proposition 2.3. *Consider the value functions $\underline{v}$, $\bar{v} : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$,*

$$\underline{v}(x,y) = \sup_{\mathcal{A}_{z_1}} E \int_0^{+\infty} e^{-\rho t} U(\underline{C}_t^{x}, \underline{C}_t^{y}) dt \tag{2.15}$$

and

$$\bar{v}(x,y) = \sup_{\mathcal{A}_{z_2}} E \int_0^{+\infty} e^{-\rho t} U(\bar{C}_t^{\bar{x}}, \bar{C}_t^{\bar{y}}) dt. \tag{2.16}$$

Then

$$\underline{v}(x,y) \le V(x,y;z) \le \bar{v}(x,y)$$

for $(x,y) \in [0,+\infty] \times [0,+\infty)$ and $z = z_1, z_2$.

We finish this section by discussing two collective-utility asset pricing problems without political risk but with different discount factors and "enhanced" collective-bequest functions. It turns out, as it is stated in Proposition 2.4 that their value functions u^1 and u^2 — given below by (2.19) and (2.20) — coincide with the original value function for states z_1 and z_2. To this end, we consider two collective-bequest functions $\Phi_1,\ \Phi_2 : [0,+\infty)\times[0,+\infty)\to[0,+\infty)$ given by

$$\Phi_1(x,y) = p_{12}V(x,y;z_2), \quad \Phi_2(x,y) = p_{21}V(x,y;z_1) \tag{2.17}$$

and the discount factors

$$\rho_1 = \rho + p_{12}, \quad \rho_2 = \rho + p_{21}. \tag{2.18}$$

Also, we consider the controlled state processes $(\underline{x}_t, \underline{y}_t)$ and $(\bar{x}_t, \bar{y}_t)$ solving (2.11) to (2.14) and the associated sets of admissible policies $\mathcal{A}_{z_1}$ and $\mathcal{A}_{z_2}$.

Proposition 2.4. *Define the value functions* $u^1,\ u^2 : [0,+\infty)\times[0,+\infty)\to[0,+\infty)$ *by*

$$u^1(x,y) = \sup_{\mathcal{A}_{z_1}} E\int_0^{+\infty} e^{-\rho_1 t}[U(\underline{C}_t^x, \underline{C}_t^y) + \Phi_1(\underline{x}_t, \underline{y}_t)]dt \tag{2.19}$$

with $\underline{x}_0 = x,\ \underline{y}_0 = y$*, and*

$$u^2(x,y) = \sup_{\mathcal{A}_{z_1}} E\int_0^{+\infty} e^{-\rho_2 t}[U(\overline{C}_t^{\overline{x}}, \overline{C}_t^{\overline{y}}) + \Phi_2(\bar{x}_t, \overline{y}_t)]dt \tag{2.20}$$

with $\bar{x}_0 = x,\ \overline{y}_0 = y$ *and* $\rho_1, \rho_2, \Phi_1, \Phi_2$*, as in (2.18) and (2.17). Then*

$$V(x,y;z_1) = u^1(x,y) \quad \text{and} \quad V(x,y;z_2) = u^2(x,y) \tag{2.21}$$

Idea of the proof:. Observe that $V(x,y;z_1)$ is the unique solution of the HJB equation (2.7), which can be rewritten as

$$\begin{aligned}
\min\big\{&\rho_1 V(x,y;z_1) - \mathcal{L}V(x,y;z_1) - H(V_x(x,y;z_1),\ V_y(x,y;z_1)) \\
&- z_1 x V_x(x,y;z_1) - byV_y(x,y;z_1) - \Phi_1(x,y), \\
&(1+\lambda)V_x(x,y;z_1) - V_y(x,y;z_1), -(1-\mu)V_x(x,y;z_1) + V_y(x,y;z_1)\big\} = 0.
\end{aligned} \tag{2.22}$$

We easily get that the above equation can be interpreted as the HJB equation of a new stochastic control problem with value function u^1. The result then follows from the uniqueness of (constrained) viscosity solutions of (2.7) and the fact that u^1 is a (constrained) viscosity solution of (2.22). The same type of arguments yield the result for the state z_2.

Our ultimate goal, besides understanding the behavior of the optimal shipping policies is to specify the equilibrium prices of goods **X** and **Y**. Note that we can interpret the ratio of the partial derivatives of $V(x, y; z)$ as the relative price of good **X** in terms of good **Y**. Then, the equilibrium prices for the states z_1 and z_2 will be respectively

$$P_1(x, y; z_1) = \frac{V_x(x, y; z_1)}{V_y(x, y; z_1)} \quad \text{and} \quad P_2(x, y; z_2) = \frac{V_x(x, y; z_2)}{V_y(x, y; z_2)}$$

If we further assume that $U(C^x, C^y) = (C^x)^\gamma + (C^y)^\gamma$, then each of the value functions of the two pseudo-problems (2.19) and (2.20) is homogeneous of degree γ. For the state z_1, there will be a cone with linear boundaries in the (x, y) plane within which no shipping occurs. Similarly, there will be a "no-shipping cone" for the state z_2; however, these cones will generally be different. One can observe that for the "politically favorable" state z_2 the expected returns for production in country $\mathcal{X}$ is relatively high as compared with the situation for the politically "unfavorable" state z_1. Consequently, a shift to z_1 makes production in country $\mathcal{X}$ less attractive and may cause a costly shipment to take place between countries. In other words, transitioning from z_2 to z_1 (or vice versa) can cause the no-shipping cone to shift. Hence, jumps in the coefficients of the asset prices can occur when z_t switches values; this contrasts with the smoothing changing prices obtained in Dumas (1992). On the other hand, as in Dumas (1992), the prices are expected to deviate from a parity value of one for potentially substantial periods of time. Both prices are bounded between the $(1 + \lambda)^{-1}$ and $(1 - \mu)^{-1}$ which represent the prices at the cone boundary where shipments take place. However, shifts in the cone as z_t

transitions between z_1 and z_2 can result in cone rotation. These observations are further developed in section 4 after we obtain the numerical results.

3. Numerical Schemes

This section is devoted to the construction of numerical schemes for the solution of the Variational Inequalities (2.7) and (2.8). Besides computing the value function V, we also compute the equilibrium prices $P_1(x, y; z_1)$ and $P_2(x, y; z_2)$ and the location of the free boundaries related to the optimal lump-sum shipments. Finally, we study how the presence of political risk influences the trading policies and equilibrium prices by also examining the model in the absence of political uncertainty.

The first goal in choosing the appropriate class of schemes is to find a scheme with three key properties: *consistency, monotonicity* and *stability*. We define these properties below; we use a generic notation for our equation in order to simplify the presentation.

To this end, we consider a nonlinear equation $F(w, u(w), Du(w), D^2u(w)) = 0$ for $w \in \overline{\Omega}$, where Du and D^2u denote respectively the gradient and the second order derivative matrix of the solution u; F is continuous in all its arguments and the equation is degenerate elliptic meaning that $F(w, p, q, A + B) \leq F(w, p, q, A)$ if $B \geq 0$.

Definition 3.1: *We consider a sequence of approximations* $S : \mathbf{R}^+ \times \overline{\Omega} \times \mathbf{R} \times \mathcal{B}_{\mathrm{loc}}(\overline{\Omega}) \to \mathbf{R}$ *where* $S = S(\theta, w, u^\theta(w), u^\theta)$.

We say that S is:

monotone if

$$S(\theta, w, t, u) \leq S(\theta, w, t, v) \quad \text{for } u \geq v,$$

consistent if

$$\limsup_{(\theta,y,\xi)\to(0,w,0)} \frac{S(\theta,w,\phi(w)+\xi,\phi+\xi)}{\theta} =$$

$$= \liminf_{(\theta,y,\xi)\to(0,w,0)} \frac{S(\theta,w,\phi(w)+\xi,\phi+\xi)}{\theta} = F(w,\phi(w),D\phi(w),D^2\phi(w)),$$

stable if

$\forall\theta > 0$, there exists a solution $u^\theta \in \mathcal{B}_{\mathrm{loc}}(\overline{\Omega})$ of $S(\theta,w,u^\theta(w),u^\theta) = 0$ and its (local) bound is independent of θ.

The motivation to use such schemes for our model comes from the fact that they exhibit excellent convergence properties to the (viscosity) solution of fully nonlinear degenerate elliptic partial differential equations as long as the latter have a unique solution. This result was established by Barles and Souganidis (1991)* and it is stated below for completeness.

Theorem 3.1 (Barles and Souganidis): *Assume that the equation $F(w,u,Du,D^2u) = 0$ admits the strong uniqueness property, i.e. if u (resp. v) is a viscosity subsolution (resp. supersolution) of $F = 0$, then $u \le v$. If the approximation sequence $\{S^\theta\}$ satisfies the monotonicity, consistency and stability properties then the solution u^θ of $S(\theta,w,u^\theta(w),u^\theta) = 0$ converges locally uniformly to the unique viscosity solution of $F(w,u,Du,D^2u) = 0$.*

We continue with the description of our scheme. To this end, we first write (2.7) and (2.8) in the concise form

$$\min\{\rho V - \mathcal{L}V - \mathcal{L}_0 V,\ \mathcal{L}_1 V,\ \mathcal{L}_2 V\} = 0 \tag{3.1}$$

* The variational inequalities (2.7) and (2.8) belong to the class of equations that Barles and Souganidis (1991) examined. Our problem though is not entirely identical to theirs due to the presence of the state constraints (2.5). The convergence of our scheme, in the presence of the state constraints is not presented here.

where for $i = 1, 2$ at (x, y, z_i)

$$\mathcal{L}V(x,y;z_i) = \frac{1}{2}\sigma_1^2 x^2 V_{xx} + \delta\sigma_1\sigma_2 xyV_{xy} + \frac{1}{2}\sigma_2^2 y^2 V_{yy} + z_i xV_x + byV_y$$

$$+ \max_{C^x, C^y} \big\{ -C^x V_x - C^y V_y + U(C^x, C^y) \big\}$$

with

$$\mathcal{L}_0 V(x,y;z_1) = p_{12}(V(x,y;z_2) - V(x,y;z_1)),$$

$$\mathcal{L}_0 V(x,y;z_2) = p_{21}(V(x,y;z_1) - V(x,y;z_2)),$$

$$\mathcal{L}_1 V(x,y;z_i) = (1+\lambda)V_x(x,y;z_i) - V_y(x,y;z_i),$$

$$\mathcal{L}_2 V(x,y;z_i) = -(1-\mu)V_x(x,y;z_i) + V_y(x,y;z_i).$$

The first step consists of approximating the equation in the whole space by an equation set in a bounded domain $\mathcal{B}_R = [0, R] \times [0, R]$ and proving the existence of a solution V_R of the Variational Inequalities in B_R and the convergence of V_R to V as R tends to the infinity. As there is no natural condition satisfied at infinity by $V(x, y; z_1)$ and $V(x, y; z_2)$, we have to decide what kind of condition we impose on ∂B_R. Barles, Daher and Romano (1995) answered to this question and exhibited an exponential rate of convergence for the heat equation complemented either with Dirichlet or Neumann conditions. The generalization of their result to more general parabolic equations is straightfoward (for more details, see Barles, Daher and Romano (1995)). In the degenerate elliptic case, there is no natural choice for the Dirichlet or Neumann boundary value.

We impose here a simple, arbitrary Neumann condition $\frac{\partial V_R}{\partial n}(x, y; z) = K$ where n is the outer unit vector and K is a preassigned positive constant. Note that this condition must be taken in the viscosity sense and that the corners of $\mathcal{B}_R$ require a specific treatment.

The second step is the approximation to the solution of the equation set in the above bounded domain. We denote by Δx and Δy, respectively, the mesh sizes in the x and y directions. Moreover $x_i = i\Delta x$, $y_j = j\Delta y$ are the grid points and

$V^1_{i,j}$ (resp. $V^2_{i,j}$) are the approximations for the value function $V(x,y;z_1)$ (resp. $V(x,y;z_2)$) at the grid point (x_i,y_j). We then propose an iterative algorithm to compute $V^1_{i,j}$ and $V^2_{i,j}$. For this purpose, we introduce a time step Δt and the approximation for $V^1_{i,j}$ (resp. $V^2_{i,j}$) at step n will be denoted by $V^{1,n}_{i,j}$ (resp. $V^{2,n}_{i,j}$). If $(V^{1,n}_{i,j}, V^{2,n}_{i,j})$ is known at step n, the monotone scheme which allows us to compute at step $n+1$, $(V^{1,n+1}_{i,j} V^{2,n+1}_{i,j})$ may be ultimately written as

$$\begin{aligned}
&S^1(\Delta t, \Delta x, \Delta y, n\Delta t, x_i, y_j, V^{1,n+1}_{i,j}, V^{2,n}_{i,j}, V^{1,n}_{i,j}, V^{1,n}_{i-1,j-1}, V^{1,n+1}_{i-1,j-1}, V^{1,n}_{i-1,j},\\
&\quad V^{1,n+1}_{i-1,j}, V^{1,n}_{i-1,j+1}, V^{1,n+1}_{i-1,j+1}, V^{1,n}_{i,j-1}, V^{1,n+1}_{i,j-1}, V^{1,n}_{i,j+1}, V^{1,n+1}_{i,j+1}, V^{1,n}_{i+1,j-1},\\
&\quad V^{1,n+1}_{i+1,j-1}, V^{1,n}_{i+1,j}, V^{1,n+1}_{i+1,j}, V^{1,n}_{i+1,j+1}, V^{1,n+1}_{i+1,j+1}) = 0,
\end{aligned}$$

and

$$\begin{aligned}
&S^2(\Delta t, \Delta x, \Delta y, n\Delta t, x_i, y_j, V^{2,n+1}_{i,j}, V^{1,n}_{i,j}, V^{2,n}_{i,j}, V^{2,n}_{i-1,j-1}, V^{2,n+1}_{i-1,j-1}, V^{2,n}_{i-1j},\\
&\quad V^{2,n+1}_{i-1,j}, V^{2,n}_{i-1,j+1}, V^{2,n+1}_{i-1j+1}, V^{2,n}_{i,j-1}, V^{2,n+1}_{i,j-1}, V^{2,n}_{i,j+1}, V^{2,n+1}_{i,j+1}, V^{2,n}_{i+1,j-1},\\
&\quad V^{2,n+1}_{i+1,j-1}, V^{2,n}_{i+1,j}, V^{2,n+1}_{i+1,j}, V^{2,n}_{i+1,j+1}, V^{2,n+1}_{i+1,j+1}) = 0.
\end{aligned}$$

Both S^1 and S^2 are consistent with (2.7),(2.8) as $\Delta t, \Delta x, \Delta y$ converge to 0 and $n\Delta t$ converges to $+\infty$, satisfy the monotonicity and stability property defined in (3.1). Note that $(\Delta t, \Delta x, \Delta y, n\Delta t)$ correspond to the variable θ in Definition 3.1, whereas (x_i, y_j) stands for w and $V^{1,n+1}_{i,j}$ in S^1 (resp. $V^{2,n+1}_{i,j}$ in S^2) represents $u^\theta(w)$. Finally, the role of the variable u^θ is played here for S^1 by

$$\begin{aligned}
&\Big(V^{2,n}_{i,j}, V^{1,n}_{i,j}, V^{1,n}_{i-1,j-1}, V^{1,n+1}_{i-1,j-1}, V^{1,n}_{i-1,j}, V^{1,n+1}_{i-1,j}, V^{1,n}_{i-1,j+1}, V^{1,n+1}_{i-1,j+1}, V^{1,n}_{i,j-1}, V^{1,n+1}_{i,j-1},\\
&\quad V^{1,n}_{i,j+1}, V^{1,n+1}_{i,j+1}, V^{1,n}_{i+1,j-1}, V^{1,n+1}_{i+1,j-1}, V^{1,n}_{i+1,j}, V^{1,n+1}_{i+1,j}, V^{1,n}_{i+1,j+1}, V^{1,n+1}_{i+1,j+1}\Big).
\end{aligned}$$

We continue with the description of the scheme. First, we define the following explicit approximation to the gradient operators $\mathcal{L}_1 V$ and $\mathcal{L}_2 V$

$$\frac{V^{n+1}_{i,j} - V^n_{i,j}}{\Delta t} = -(1+\lambda)\frac{V^n_{i,j} - V^n_{i-1,j}}{\Delta x} + \frac{V^n_{i,j+1} - V^n_{i,j}}{\Delta y}.$$

$$\frac{V^{n+1}_{i,j} - V^n_{i,j}}{\Delta t} = (1-\mu)\frac{V^n_{i+1,j} - V^n_{i,j}}{\Delta x} - \frac{V^n_{i,j} - V^n_{i,j-1}}{\Delta y},$$

where $V_{i,j}^n$ stands for both $V_{i,j}^{1,n}$ and $V_{i,j}^{2,n}$. It is easy to verify that these approximations are monotone as long as $\Delta t \leq \frac{min(\Delta x,\Delta y)}{2+\lambda}$.

For the elliptic operator $\mathcal{L}$, we use a *time-splitting method* in order to approximate separately the first-order derivatives in a first-half iteration and the second-order ones in the second-half iteration.

For the first-half iteration, we consider the *first-order operator* $\tilde{\mathcal{L}}$ obtained by eliminating the second-order terms in $\mathcal{L}$, i.e., for $i = 1, 2$,

$$\tilde{\mathcal{L}}V(x, y; z_i) = z_i x V_x + byV_y + \max_{C^x, C^y} \{ - C^x V_x - C^y V_y + U(C^x, C^y)\}.$$

The solution to the equation $\rho\tilde{V} - \tilde{\mathcal{L}}\tilde{V}(x, y; z_i) - \mathcal{L}_0\tilde{V}(x, y; z_i) = 0$ can be characterized (see for example Lions (1983)) as the value function of a stochastic control problem

$$\tilde{V}(x, y; z) = \sup_{\tilde{\mathcal{A}}} E\Big\{ \int_0^{+\infty} e^{-\rho t} U(\tilde{C}_t^x, \tilde{C}_t^y) dt \Big\} \tag{3.2}$$

where the state trajectories $\tilde{x}_t, \tilde{y}_t$ solve

$$\begin{cases} d\tilde{x}_t = z_i \tilde{x}_t dt - \tilde{C}_t^x dt - (1+\lambda) d\tilde{L}_t + (1-\mu) d\tilde{M}_t \\ d\tilde{y}_t = b\tilde{y}_t dt - \tilde{C}_t^y dt + d\tilde{L}_t - d\tilde{M}_t \\ \tilde{x}_0 = x, \quad \tilde{y}_0 = y \end{cases}$$

We apply the Dynamic Programming Principle to the above control problem and discretize it, that is, for Δt positive and sufficiently small, we choose a constant approximation to each consumption rate on the time interval $[0, \Delta t]$.

We then compute the optimum in closed-form in order to obtain the following numerical scheme for $\tilde{V}(x, y; z_1)$ which is monotone for Δt sufficiently small:

$$\frac{V_{i,j}^{1,n+1/2} - V_{i,j}^{1,n}}{\Delta t} = p_{12}(V_{i,j}^{2,n} - V_{i,j}^{1,n}) - \rho V_{i,j}^{1,n} +$$

$$h_1(\Delta x, x_i, V_{i-1,j}^{1,n}, V_{i,j}^{1,n}, V_{i+1,j}^{1,n}) + h_2(\Delta y, y_j, V_{i,j-1}^{1,n}, V_{i,j}^{1,n}, V_{i,j+1}^{1,n})$$

where h_1, in the case of a CRRA utility with $\gamma = 0.5$, is defined by:

$$h_1(\Delta x, x_i, V^{1,n}_{i-1,j}, V^{1,n}_{i,j}, V^{1,n}_{i+1,j}) = \frac{\Delta x}{V^{1,n}_{i+1,j} - V^{1,n}_{i,j}} + z_1 x_i \frac{V^{1,n}_{i+1,j} - V^{1,n}_{i,j}}{\Delta x}$$

if $z_1 x_i \geq \frac{\Delta x}{(V^{1,n}_{i+1,j} - V^{1,n}_{i,j})^2}$ and $z_1 x_i \geq \frac{\Delta x}{(V^{1,n}_{i,j} - V^{1,n}_{i-1,j})^2}$,

$$h_1(\Delta x, x_i, V^{1,n}_{i-1,j}, V^{1,n}_{i,j}, V^{1,n}_{i+1,j}) = \frac{\Delta x}{V^{1,n}_{i,j} - V^{1,n}_{i-1,j}} + z_1 x_i \frac{V^{1,n}_{i,j} - V^{1,n}_{i-1,j}}{\Delta x}$$

if $(z_1 x_i \geq \frac{\Delta x}{(V^{1,n}_{i+1,j} - V^{1,n}_{i,j})^2}$ and $z_1 x_i < \frac{\Delta x}{(V^{1,n}_{i,j} - V^{1,n}_{i-1,j})^2})$ or
$(z_1 x_i < \frac{\Delta x}{(V^{1,n}_{i+1,j} - V^{1,n}_{i,j})^2}$ and $z_1 x_i < \frac{\Delta x}{(V^{1,n}_{i,j} - V^{1,n}_{i-1,j})^2})$

and

$$h_1(\Delta x, x_i, V^{1,n}_{i-1j}, V^{1,n}_{i,j}, V^{1,n}_{i+1,j}) = 2\sqrt{z_1 x_i},$$

if $z_1 x_i < \frac{\Delta x}{(V^{1,n}_{i+1,j} - V^{1,n}_{i,j})^2}$ and $z_1 x_i \geq \frac{\Delta x}{(V^{1,n}_{i,j} - V^{1,n}_{i-1,j})^2}$.

Symmetrically h_2 is deduced from h_1 by replacing respectively Δx, $z_1 x_i$,$V^{1,n}_{i-1,j}$ and $V^{1,n}_{i+1,j}$ by $\Delta y, by_j, V^{1,n}_{i,j-1}$ and $V^{1,n}_{i,j+1}$ and the approximation $V^{2,n}_{i,j}$ is obtained similarly.

A simple sufficient condition for the monotonicity of the previous approximation is provided by the following upper bound on the time-step

$$\Delta t \leq \min_{k \in \{1,2\}, i, j} \left\{ \frac{1}{z_2 i + bj + \max(p_{12}, p_{21}) + \rho + \frac{\Delta x}{(V^{k,n}_{i,j} - V^{k,n}_{i-1,j})^2} + \frac{\Delta y}{(V^{k,n}_{i,j} - V^{k,n}_{i,j-1})^2}} \right\}.$$

The *second order degenerate elliptic term* is then approximated by the well-known Crank-Nicolson scheme. To simplify the presentation, we chose the following approximation for the second-order derivatives which in fact is not monotone but the replacement by a monotone approximation is routine and this modification does not affect the convergence of the scheme. It is worth mentioning that this scheme is unconditionally stable, independently of Δt, which may be chosen large. As before, we use the notation $V^n_{i,j}$ for both $V^{1,n}_{i,j}$ and $V^{2,n}_{i,j}$.

$$\frac{V_{i,j}^{n+1}-V_{i,j}^{n+1/2}}{\Delta t}=\frac{1}{2}\sigma_1^2x_i^2\Big[\frac{1}{2}\frac{(V_{i+1,j}^{n+1/2}+V_{i-1,j}^{n+1/2}-2V_{i,j}^{n+1/2})}{\Delta x^2}+$$

$$\frac{1}{2}\frac{(V_{i+1,j}^{n+1}+V_{i-1,j}^{n+1}-2V_{i,j}^{n+1})}{\Delta x^2}\Big]+\frac{1}{2}\sigma_2^2y_j^2\Big[\frac{1}{2}\frac{(V_{i,j+1}^{n+1/2}+V_{i,j-1}^{n+1/2}-2V_{i,j}^{n+1/2})}{\Delta y^2}+$$

$$\frac{1}{2}\frac{(V_{2i,j+1}^{n+1}+V_{i,j-1}^{n+1}-2V_{i,j}^{n+1})}{\Delta y^2}\Big]+$$

$$\delta\sigma_1\sigma_2x_iy_j\Big[\frac{1}{2}(\frac{V_{i+1,j+1}^{n+1/2}+V_{i-1,j-1}^{n+1/2}-V_{i-1,j+1}^{n+1/2}-V_{i+1,j-1}^{n+1/2}}{4\Delta x\Delta y})+$$

$$\frac{1}{2}(\frac{V_{i+1,j+1}^{n+1}+V_{i-1,j-1}^{n+1}-V_{i+1,j-1}^{n+1}-V_{i-1,j+1}^{n+1}}{4\Delta x\Delta y})\Big].$$

On the x-axis, we impose for $V_{i,j}^1$ and $V_{i,j}^2$ the gradient condition in the following format

$$\frac{V_{i,j}^{n+1}-V_{i,j}^n}{\Delta t}=-(1+\lambda)\frac{V_{i,j}^n-V_{i-1,j}^n}{\Delta x}+\frac{V_{i,j+1}^n-V_{i,j}^n}{\Delta y}.$$

Similarly, on the y-axis, we impose

$$\frac{V_{i,j}^{n+1}-V_{i,j}^n}{\Delta t}=(1-\mu)\frac{V_{i+1,j}^n-V_{i,j}^n}{\Delta x}-\frac{V_{i,j}^n-V_{i,j-1}^n}{\Delta y}.$$

At each iteration, we choose the following adaptative time-step which actually is not far from being constant but may evolve a little during the convergence:

$$\Delta t=\min\Big\{\min_{k\in\{1,2\},i,j}\Big\{\frac{1}{z_2i+bj+\max(p_{12},p_{21})+\rho+\frac{\Delta x}{(V_{i,j}^{k,n}-V_{i-1j}^{k,n})^2}+\frac{\Delta y}{(V_{i,j}^{k,n}-V_{ij-1}^{k,n})^2}}\Big\},$$

$$\frac{\min(\Delta x,\Delta y)}{2+\lambda}\Big\}.$$

Given the approximations to the elliptic and the gradient operators, $V_{i,j}^{1,n}$ is then set to the maximal value over these three ones. Futhermore, we let the algorithm converge until the conditions $\sup_{i,j}|V_{i,j}^{1,n}-V_{i,j}^{1,n-1}|<\epsilon$ and $\sup_{i,j}|V_{i,j}^{2,n}-V_{i,j}^{2,n-1}|<\epsilon$ are reached, ϵ being a preassigned small positive constant. After the last iteration, we compute the equilibrium prices by using centered finite differences and finally, the no-shipping region is defined as the set of the points where the approximation to the value function at the last step comes from the discretization of the elliptic operator.

We continue with a brief description of the numerical experiments: let $\Delta x = \Delta y = 0.1$, $R = 10$ and $\epsilon = 5.10^{-5}$. We chose the following parameters: $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\lambda = \mu = 0.05$, $\rho = 0.1$, $\gamma = 0.5$, $\delta = 0.5$ and $b = 0.1$.

Figures 1–4 show the no-shipping regions and the equilibrium prices for the states z_1 and z_2 in the absence of political uncertainty. More precisely, Figures 1 and 2 correspond to $z_1 = z_2 = 0.1$ with $p_{12} = p_{21} = 0$ while figures 3 and 4 correspond to $z_1 = z_2 = 0.08$ with $p_{12} = p_{21} = 0$.

Then, in order to study the influence of the transition probabilities, in Figures 5–12 are represented the no-shipping regions for the states $z_1 = 0.08$ and $z_2 = 0.1$ respectively in the following four cases

Case A: $p_{12} = p_{21} = 0.1$

Case B: $p_{12} = 0.1$, $p_{21} = 0.9$

Case C: $p_{12} = 0.9$, $p_{21} = 0.1$

Case D: $p_{12} = 0.9$, $p_{21} = 0.9$

Finally, in Figures 13–16 we graph the equilibrium prices for the above four cases.

The scheme does not behave in a perfectly stable way at least for the no-shipping region. If one lets the scheme converge for a very long time, the cone remains globally the same, except that a few points which oscillate around the free boundaries, that is, they appear and disappear from iteration to iteration. Apparently, this phenomenon might be caused by possibly over-estimated Neumann conditions for large values of x and y.

4. Concluding remarks

In this paper, we have developed a model of international asset pricing in the presence of political risk. Although the model is considerably simplified, it represents a substantial step towards understanding how uncertainty about future government actions can affect the prices of tradeable assets. The recent turmoil in asset prices for several Southeast Asian countries serves to emphasize the importance of gaining a better understanding of the effects of political risk on asset prices.

Our numerical experiments with the model provide several interesting results. As in Dumas (1992), we obtain a cone in the state space within which no shipping occurs. That is, individuals find it optimal not to incur the shipping costs entailed with adjusting their relative holdings of the two assets whenever their asset positions lie within this cone. In our model, the size and location of this cone depend on the political state in the Country $\mathcal{X}$ (the politically risky country). Consider, for example, the figures 5 and 6. In figure 5, Country $\mathcal{X}$ is in the poor political state with relatively low expected returns on asset X. In figure 6, Country $\mathcal{X}$ has now switched to the favorable political state. Asset X is now more valuable and individuals are less inclined to export from Country $\mathcal{X}$. They are also more inclined to pay the shipping cost to import from Country $\mathcal{Y}$ and, in effect, convert some of their position in asset Y into asset X. These two changes in their relative willingness to trade are manifested in the downward (clockwise) rotation of the no-shipping cone between figures 5 and 6.

We can also see that the transition probabilities influence the rotation and size of the non-shipping cone. Compare, for example, the figures 3 and 5. For both figures, Country $\mathcal{X}$ is in the poor political state. However, in figure 5 there is a 10% probability of transitioning to the better state whereas in figure 3 that transition probability is zero. Intuitively, the increased probability of moving to a better state increases the value of asset X and alters individuals' willingness to trade. In this case, the primary effect is a reduced willingness to export X which results in a downward rotation in the lower boundary of the no-shipping cone.

We also provide graphical comparisons on the relative prices of goods $\mathbf{X}$ and

$\mathbf{Y}$. Consider, for example, figure 13. In this figure, the quantity of $\mathbf{X}$ is fixed while the quantity of $\mathbf{Y}$ is varied and the relative price of $\mathbf{X}$ (in terms of $\mathbf{Y}$) is plotted in each of the two political states. In state z_2 (the favorable political states) asset X is relatively more valuable. However, it is interesting to note that for situations where asset Y is either quite scarce or extremely plentiful, the political state seems to have a negligible effect on relative asset pricing. Intuitively, when Y is very scarce, its value becomes extremely high and the relative value of X becomes so small that the effect of differing political states is not apparent. A symmetric argument applies when Y is extremely plentiful. Comparing, for example, figures 13 and 14 we can see that the transition probabilities have a substantial effect on the extent to which the political state alters the relative pricing of $\underline{\mathcal{X}}$ and $\underline{\mathcal{Y}}$. In figure 16, the transition probabilities have both become so great that the relative price difference across political states all but disappears.

In conclusion, we view the implications of political risk for asset pricing as both interesting and of considerable economic importance. This paper represents a step towards better understanding some of those implications, and we hope that the paper will stimulate further research on this important issue.

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Appendix A

Consider a non-linear second order partial differential equation of the form

$$F(X, \bar{v}, D\bar{v}, D^2\bar{v}) = 0 \quad \text{in } \Omega, \tag{A.1}$$

where $D\bar{v}$ and $D^2\bar{v}$ stand respectively for the gradient vector and the second derivative matrix of $\bar{v}$; F is continuous in all its arguments and degenerate elliptic, meaning that

$$F(X, p, q, A + B) \leq F(X, p, q, A) \quad \text{if } B \geq 0. \tag{A.2}$$

Definition A.1. *A continuous function $u : R \to R$ is a constrained viscosity solution of (A.1) if*

i) u is a viscosity subsolution of (A.1) on $\overline{\Omega}$, that is for any $\phi \in C^2(\overline{\Omega})$ and any local maximum point $X_0 \in \overline{\Omega}$ of $u - \phi$

$$F(X_0, u(X_0), D\phi(X_0), D^2\phi(X_0)) \leq 0$$

and

ii) u is a viscosity supersolution of (A.1) in Ω, that is for any $\phi \in C^2(\overline{\Omega})$ and any local minimum point $X_0 \in \Omega$ of $u - \phi$

$$F(X_0, u(X_0), D\phi(X_0), D^2\phi(X_0)) \geq 0.$$

Appendix B

Proof of Proposition 2.2: We are only going to prove the proposition for the case $b > z_2$ and $\hat{\rho}_1 > 0$, $k_1 > 0$, together with the first of inequalities (2.6b), since the other case can be worked out along the same lines.

We first recall Proposition 2.3 which states that the value function V is bounded from above by the value function $\bar{v}$ of the same stochastic control problem when there is no political risk and the process $z_t = z_2$, with z_2 being the higher state.

The rest of the proof amounts to demonstrating that $\bar{v}(x, y)$ is bounded from above by a value function which corresponds to the standard model of portfolio model with transaction costs, for which problem conditions (2.6a) and (2.6b) apply (see Davis and Norman (1987) and Shreve and Soner (1994)). To this end, observe that using the growth condition for the utility function U and a suboptimal in general policy (related to the gradient constraint $\bar{v}_y \geq (1 - \mu)\bar{v}_x$), we have that

$$\bar{v}(x, y) \leq \bar{V}(x, y)$$

where $\bar{V}$ is the value function of the following stochastic control problem.

Consider the state equations

$$d\tilde{x}_t = z_2\tilde{x}_t dt + \sigma_1\tilde{x}_t dW_t^1 - \tilde{C}_t dt - (1+\lambda)d\tilde{L}_t + (1-\mu)d\tilde{M}_t$$
$$d\tilde{y}_t = b\tilde{y}_t dt + \sigma_2\tilde{y}_t dW_t^2 + d\tilde{L}_t - d\tilde{M}_t$$

and payoff

$$J(x,y,\tilde{C},\tilde{L},\tilde{M}) = E\int_0^{+\infty} e^{-\rho t} M\tilde{C}_t^{\gamma} dt.$$

For the rest of the arguments, we set $M = 1$ and use $\mathcal{A}$ as the generic set of admissible policies defined along the same lines as for the original problem. Then we define

$$\bar{V}(x,y) = \sup_{\mathcal{A}} E\int_0^{+\infty} e^{-\rho t}\tilde{C}_t^{\gamma} dt.$$

It is immediate to verify, using the power form of the utility and the linearity of the above state equations, that $\bar{V}$ is homogeneous of degree γ. In other words, $\bar{V}(x,y) = x^{\gamma}F(w)$ with $w = \frac{y}{x}$ and F solving the Variational Inequality

$$\min\Big\{\hat{\rho}_1 F - \frac{1}{2}\sigma^2 w^2 F_{ww} - k_1 w F_w - \max_{c\geq 0}\big\{ - cF_w + c^{\gamma}\big\},$$
$$\gamma(1+\lambda)F + F_w(1-(1+\lambda)w), -\gamma(1-\mu)F + F_w(1+(1-\mu)w)\Big\} = 0. \quad \text{(B.1)}$$

Now, consider the standard model of portfolio management in markets with transaction costs as in Davis and Norman (1987). In their model, there are two assets: a bond with riskless rate r, and a stock with mean rate of return μ and volatility σ. The utility function is of the same power type and the discount factor is β. In order to have a well-defined value function, Davis and Norman (1987) imposed the condition $\beta \geq r\gamma + (\gamma(\mu - r)^2/2\sigma^2(1-\gamma))$. For the same problem, Shreve and Soner (1994) provided a different set of conditions, in that

$$\beta > r\gamma + \frac{\gamma^2(\mu - r)^2}{2\sigma^2(1-\gamma)^2}.$$

Comparing coefficients with (B.1) and after some tedious but straightforward otherwise calculations we see that in order for F to be finite – and therefore the value function V – we must have either

$$\hat{\rho}_1 \geq z_2(\gamma - 1) + \frac{\gamma k_1^2}{2\sigma^2(1-\gamma)}$$

or

$$\hat{\rho}_1 \geq z_2(\gamma - 1) + \frac{\gamma^2 k_1^2}{2\sigma^2(1-\gamma)^2}.$$

Proof of Theorem 2.1: The fact that the value function is a constrained viscosity solution of the system of Variational Inequalities (2.7) and (2.8) follows from a combination of the arguments used in Zariphopoulou (1991) and in Tourin and Zariphopoulou (1994).

In order to establish that the value function is the unique constrained viscosity solution, we need to construct a positive strict supersolution for (2.7) and (2.8). Once this supersolution is found, the rest of the arguments are similar to the ones used in Tourin and Zariphopoulou (1994) and they are not presented herein. We continue with the construction of the positive strict supersolution of (2.7) and (2.8), i.e. a function, say $G(x,y;z)$ such that

$$\begin{aligned}\min\Big\{ &\min\big\{\rho G(x,y;z_1) - \mathcal{L}G(x,y;z_1) - H(G_x(x,y;z_1), G_y(x,y;z_1))\\ &- p_{12}(G(x,y;z_2) - G(x,y;z_1)) - z_1 x G_x(x,y;z_1)\\ &- byG_y(x,y;z_1),\ \ (1+\lambda)G_x(x,y;z_1) - G_y(x,y;z_1),\\ &-(1-\mu)G_x(x,y;z_1) + G_y(x,y;z_1)\big\},\\ &\min\big\{\rho G(x,y;z_2) - \mathcal{L}G(x,y;z_2) - H(G_x(x,y;z_2), G_y(x,y;z_2))\\ &- p_{21}(G(x,y;z_1) - G(x,y;z_2)) - z_2 x G_x(x,y;z_2)\\ &- byG_y(x,y;z_2),\ \ (1+\lambda)G_x(x,y;z_2) - G_y(x,y;z_2),\\ &-(1-\mu)G_x(x,y;z_2) + G_y(x,y;z_2)\big\}\Big\} > \theta \end{aligned} \tag{B.2}$$

for some positive constant θ.

To this end, we claim that there is an increasing in (x,y) and independent of z function $G(x,y)$ such that the above inequality holds. In fact, first observe that for such a function it suffices to show that

$$\begin{aligned}\min\Big\{&\rho G - \mathcal{L}G - z_2 x G_x - byG_y - H(G_x, G_y),\\ &(1+\lambda)G_x - G_y, -(1-\mu)G_x + G_y\Big\} > \theta.\end{aligned} \tag{B.3}$$

We continue with the assumption that $b > z_2$; the case $b \le z_2$ can be worked out using similar arguments. Since H is a decreasing function of its arguments, if G satisfies $G_y > (1-\mu)G_x$, then, in order to establish (B.3), it suffices to show

$$\begin{aligned}\min\big\{&\rho G - \mathcal{L}G - z_2 x G_x - byG_y - \max_{c\ge 0}\{-cG_x + c^\gamma\},\\ &(1+\lambda)G_x - G_y, -(1-\mu)G_x + G_y\big\} > 0.\end{aligned} \tag{B.4}$$

The above inequality follows from the properties of the utility function and the nature of $H(\ ,\ )$ as the following arguments show.

$$
\begin{aligned}
H(G_x, G_y) &= \max_{C^x, C^y} \left\{ -C^x G_x - C^y G_y + U(C^x, C^y) \right\} \\
&\leq \max_{C^x, C^y} \left\{ -(C^x + (1-\mu)C^y) G_x + U(C^x, C^y) \right\} \\
&\leq \max_{C^x, C^y} \left\{ -(C^x + (1-\mu)C^y) G_x + (C^x + (1-\mu)C^y)^\gamma \right\} \\
&= \max_{c} \left\{ -cG_x + c^\gamma \right\}
\end{aligned}
$$

where $C^x \geq 0$, $C^y \geq 0$ and $c = C^x + (1-\mu)C^y \geq 0$.

Note that if $b < z_2$, it is more convenient to use that $G_x > \frac{1}{1+\lambda} G_y$ and work with the inequality

$$
H(G_x, G_y) \leq \max_{C^x, C^y} \left\{ -\Big(\frac{1}{1+\lambda} C^x + C^y\Big) G_x + \Big(\frac{1}{1+\lambda} C^x + C^y\Big)^\gamma \right\}.
$$

The construction of a function that satisfies (B.4) was explicitly executed in Tourin and Zariphopoulou (1994) when the operator $\mathcal{L}$ is hypoelliptic in x i.e. when there are no higher than first order derivatives with respect to x. On the other hand, the general case we examine herein can be reduced to the degenerate case by manipulating the homogeneity properties of the value function.

To this end, we define the function G as follows. First we consider the solution g of

$$
\begin{cases}
(\hat{\rho}_1 + z_2) g = -\frac{k_1^2}{2\sigma^2} \frac{g_w^2}{g_{ww}} + z_2 w g_w + \max\limits_{c \geq 0} \{-c g_w + c^\gamma\} \\
g > 0, g_w > 0 \ \text{ and } \ g_{ww} < 0.
\end{cases}
$$

The reader familiar with continuous time portfolio choice problems will recognize that g is the solution to the classical Merton consumption-portfolio problem.

Now, define G by

$$
G(x, y) = g(x + ky) + K + n_1 x + n_2 y
$$

where K, n_1, n_2 and k are positive constants and n_1, n_2 and k satisfy

$$
1 - \mu < k < 1 + \lambda \ \text{ and } \ (1+\lambda) n_1 > n_2 > \frac{\hat{\rho}_1}{\hat{\rho}_1 - (b - z_2)} (1-\mu) n_2.
$$

It then follows that

$$
\begin{cases}
(1+\lambda) G_x(x,y) - G_y(x,y) = (1 + \lambda - k) g'(x + ky) + [(1+\lambda) n_1 - n_2] \\
-(1-\mu) G_x(x,y) + G_y(x,y) = (-1 + \mu + k) g'(x + ky) + [-(1-\mu) n_1 + n_2].
\end{cases}
\tag{B.5}
$$

For the second order operator we use the choice of G, the equation that g satisfies and the homogeneity of the utility function. After tedious but straightforward calculations we get that

$$\rho G - \mathcal{L}G - z_2 x G_x - by G_y - \max_{c \geq 0}\{-cG_x + c^\gamma\} \geq (\hat{\rho}_1 + z_2)K. \tag{B.6}$$

Combining (B.5) and (B.6), we see that G satisfies (B.4) with $\theta = \min\left\{(\hat{\rho}_1 + z_2)K,\ (1+\lambda)n_1 - n_2, n_2 - (1-\mu)n_1\right\}$ and that $\theta > 0$.